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Poincaré–Birkhoff–Witt theorems and generalized Casimir invariants for some infinite-dimensional Lie groups: I

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Abstract. Analogues of the Poincaré–Birkhoff–Witt theorem for several pairs of Lie groups acting on complex manifolds are studied. These results lead to algebraic analogues of a theorem of I Segal on the two-sided regular representation of a unimodular locally compact group for dual representations of some infinite-dimensional Lie groups on generalized Bargmann–Segal–Fock spaces. Generalized Casimir invariants for these dual representations are also developed.

1. Introduction

The Poincaré–Birkhoff–Witt theorem, hereafter referred to as PBW, is one of the most fundamental results in the theory of Lie algebras and Lie groups. In [Po] H Poincaré established the existence of the universal enveloping algebra of a Lie algebra and proved the so-called PBW theorem (see also [Bi, Wi]); thus introducing the principal device for converting Lie algebra problems into associative algebra problems. In the context of this paper we state the following version of PBW which is essentially the one given by Poincaré.

Let G be a Lie group and let $\mathfrak{g}$ be the Lie algebra of right (resp. left) invariant vector fields on G. Let $\mathcal{U}(\mathfrak{g})$ denote the associative algebra of right (resp. left) invariant differential operators of all orders. If $(X_i)_{1\leqslant i\leqslant n}$ (resp. $(Y_i)_{1\leqslant i\leqslant n}$) is a basis (of $\mathfrak{g}$) of right (resp. left) invariant vector fields then the ordered monomials 1 and $X_{i_1}\ldots X_{i_s}$ (resp. $Y_{i_1}\ldots Y_{i_s}$) with $i_1\leqslant i_2\leqslant\cdots\leqslant i_s$; $s\geqslant 1$, constitute a (vector space) basis for $\mathcal{U}(\mathfrak{g})$.

For a more abstract formulation of PBW in terms of the universal enveloping algebra of an abstract Lie algebra see, e.g., [Bo, Ja, Di, Va]. Our first PBW-type theorem can be formulated as follows. Set $M=\mathbb{C}^{n\times k}$, $G=GL(k,\mathbb{C})$ and $G'=GL(n,\mathbb{C})$. Then G (resp. G') acts on M by right (resp. left) matrix multiplication. If f is a holomorphic function (henceforth denoted by 'of class $C^\omega(M)$') on M we define $R(g)f$ (resp. $L(g')f$), $g\in G$, $g'\in G'$, by

$$(R(g)f)(Z)=f(Zg)\qquad\text{and}\qquad (L(g')f)(Z)=f((g')^tZ)\qquad Z\in M.$$

Then we have the following definition.

Definition 1.1. *Let $\mathcal{D}^\omega(M)$ denote the algebra of all C^ω differential operators on M (see equation (2.6) for a precise definition of $\mathcal{D}^\omega(M)$). A differential operator D of $\mathcal{D}^\omega(M)$ on M is said to be right (resp. left) invariant if $D(R(g)f)=R(g)(Df)$ (resp. $D(L(g')f)=L(g')(Df)$) for all $g\in G$ (resp. $g'\in G'$), and for all f of class $C^\omega(M)$.*

Theorem 2.1 is then a generalization of PBW for the pair (G', G); in particular, when $n = k$ it is the PBW theorem for the group $GL(n, \mathbb{C})$. Now suppose $k \geqslant 2n$ and let $M = \{Z \in \mathbb{C}^{n\times k} : ZZ^t = 0, \operatorname{rank}(Z) = n\}$, then M is a complex analytic manifold. Let $G = SO(k, \mathbb{C})$ and $G' = GL(n, \mathbb{C})$, then G and G' act on M by right and left multiplications, respectively. We then have an analogue of PBW for this pair of groups. Similarly, we have the analogues of PBW for other pairs (G', G) where G is $Sp_{2k}(\mathbb{C})$ or $GL(k, \mathbb{C}) \otimes GL^{\checkmark}(k, \mathbb{C})$ (where $GL^{\checkmark}(k, \mathbb{C})$ is the abbreviated notation for the group $GL(k, \mathbb{C})$ acting contragradiently on the manifold M) and G' is a complex general linear group (see [TT1] for details on these pairs).

Now let G be any unimodular locally compact group and let $L^2(G, \mu)$ denote the Hilbert space of functions on G square-integrable relative to the Haar measure μ. Let R (resp. L) denote the right (resp. left) regular representation of G on $L^2(G, \mu)$. The closure $\mathcal{R}$ (resp. $\mathcal{L}$) in the weak operator topology of the set of all linear combinations of the $R(g)$ (resp. $L(g)$), $g \in G$, *is the right (resp. left) weakly closed group algebra of* G. The set of all bounded operators on $L^2(G, \mu)$ that commute which each element of $\mathcal{R}$ (resp. $\mathcal{L}$) is called the *commutant of* $\mathcal{R}$ *(resp.* $\mathcal{L}$*)* and is denoted by $\mathcal{R}'$ (resp. $\mathcal{L}'$). Let $\mathcal{Z}_{\mathcal{R}}$ (resp. $\mathcal{Z}_{\mathcal{L}}$) denote the *centre* of $\mathcal{R}$ (resp. $\mathcal{L}$). In physics elements of a centre of some kind are sometimes called *generalized Casimir operators* or *Casimir invariants* (the quadratic invariant of a Lie group is the classical Casimir operator or Casimir element; for further details see, e.g., [Ba+Ra]). In [Se1] I Segal proved that

$$\mathcal{R}' = \mathcal{L}, \mathcal{L}' = \mathcal{R} \qquad \text{and} \qquad \mathcal{Z}_{\mathcal{L}} = \mathcal{Z}_{\mathcal{R}} = \mathcal{R} \cap \mathcal{L}. \tag{1.1}$$

A version of this theorem for von Neumann algebras of dual pairs was given in [Ho1] and an algebraic version for classical groups is given in [Ho2]. As a consequence of theorem 2.1, in this paper we shall give several analogues to this beautiful theorem with Bergmann–Segal–Fock spaces playing the role of $L^2(G, \mu)$, algebras of differential operators playing the role of the algebra of bounded linear operators on $L^2(G, \mu)$, dual pairs of either finite- or infinite-dimensional groups replacing the unimodular locally compact group, universal or generalized universal enveloping algebras replacing group algebras, and centralizers replacing commutants, etc. In this paper we prove these theorems for the pair $(GL(n, \mathbb{C}), GL(k, \mathbb{C}))$ and its infinite-dimensional analogues. This will enable us to prove similar theorems for other pairs of groups (G', G) which will appear in a forthcoming paper. Parts of these results were announced in [TT1, TT2].

2. A PBW theorem for the pair $(GL(n, \mathbb{C}), GL(k, \mathbb{C}))$

Recall that $G = GL(k, \mathbb{C})$, $G' = GL(n, \mathbb{C})$ and $M = \mathbb{C}^{n\times k}$. For $Z \in M$ set $\partial_{\alpha i} = \frac{\partial}{\partial Z_{\alpha i}}$, $1 \leqslant \alpha \leqslant n$, $1 \leqslant i \leqslant k$. Let $\mathrm{d}R$ (resp. $\mathrm{d}L$) denote the differential of R (resp. L) which we shall refer to as the infinitesimal action of R (resp. L). Then it can be easily shown that the sets

$$R_{ij} = \sum_{\gamma=1,\ldots,n} Z_{\gamma i}\partial_{\gamma j} \qquad \text{and} \qquad L_{\alpha\beta} = \sum_{\ell=1,\ldots,k} Z_{\alpha\ell}\partial_{\beta\ell}$$
$$1 \leqslant \alpha \qquad \beta \leqslant n \qquad 1 \leqslant i \qquad j \leqslant k \tag{2.1}$$

are infinitesimal generators. We have

$$\begin{aligned} [R_{ij}, R_{uv}] &= \delta_{ju}R_{iv} - \delta_{vi}R_{uj} \qquad 1 \leqslant i, j, u, v \leqslant k \\ [L_{\alpha\beta}, L_{\mu\nu}] &= \delta_{\beta\mu}L_{\alpha\nu} - \delta_{\nu\alpha}L_{\mu\beta} \qquad 1 \leqslant \alpha, \beta, \mu, \nu \leqslant n. \end{aligned} \tag{2.1$'$}$$

Obviously, the R_{ij} (resp. $L_{\alpha\beta}$) generate a Lie algebra isomorphic to $\mathfrak{gl}(k, \mathbb{C})$ (resp. $\mathfrak{gl}(n, \mathbb{C})$). Since $L(g')R(g)f = R(g)L(g')f$ for all $g \in G$, $g' \in G'$, and $f \in C^{\omega}(M)$ it follows that the $L_{\alpha\beta}$ (resp. R_{ij}) constitute a system of *right (resp. left) invariant vector fields on* M. Let

$\mathfrak{g}'$ (resp. $\mathfrak{g}$) denote this Lie algebra of right (resp. left) invariant vector fields. For $Z \in M$ we choose an ordering on the row and column indices of Z and write $(\alpha, \beta) \leqslant (\mu, \nu)$ (resp. $(i, j) \leqslant (u, v)$) if (α, β) precedes (μ, ν) in this ordering (resp. (i, j) precedes (u, v)).

Theorem 2.1. *Let $\mathcal{D}(\mathfrak{g}')$ (resp. $\mathcal{D}(\mathfrak{g})$) be the associative subalgebra of $\mathcal{D}^{\omega}(M)$ which consists of all right (resp. left) invariant differential operators on M.*

(i) If $n \geqslant k$ then the ordered monomials 1 and $L_{11}^{r_{11}} L_{21}^{r_{12}} \ldots L_{k1}^{r_{1k}} L_{12}^{r_{21}} \ldots L_{k2}^{r_{2k}} \ldots L_{1n}^{r_{n1}} \ldots L_{kn}^{r_{nk}}$ (resp. $R_{11}^{s_{11}} \ldots R_{kk}^{s_{kk}}$) form a basis for $\mathcal{D}(\mathfrak{g}')$ (resp. $\mathcal{D}(\mathfrak{g})$), where r_{ij} (resp. s_{ij}) are non-negative integers such that $r_{11} + r_{12} + \cdots + r_{nk} = r$, $r \geqslant 1$ (resp. $s_{11} + \cdots + s_{nn} = s$, $s \geqslant 1$).

(ii) If $n < k$ then the ordered monomials 1 and $L_{11}^{r_{11}} \ldots L_{nn}^{r_{nn}}$ (resp. $R_{11}^{s_{11}} R_{21}^{s_{12}} \ldots \ldots R_{n1}^{s_{1n}} R_{12}^{s_{21}} \ldots R_{n2}^{s_{2n}} \ldots R_{1k}^{s_{k1}} \ldots R_{nk}^{s_{kn}}$) form a basis for $\mathcal{D}(\mathfrak{g}')$ (resp. $\mathcal{D}(\mathfrak{g})$), where r_{ij} (resp. s_{ij}) are non-negative such that $r_{11} + \cdots + r_{nn} = r$, $r \geqslant 1$ (resp. $s_{11} + s_{12} + \cdots + s_{kn} = s$, $s \geqslant 1$).

Proof. By symmetry it suffices to prove the theorem for the case of right-invariant differential operators. Let $\mathcal{B}$ denote the ordered basis $\{\partial_{11}, \ldots, \partial_{1k}, \partial_{21}, \ldots, \partial_{2k}, \partial_{n1}, \ldots, \partial_{nk}\}$ of vector fields on M. Then the matrix of the ordered system $\mathcal{S}$ of vector fields $\{L_{11}, \ldots, L_{n1}, L_{12}, \ldots, L_{n2}, \ldots, L_{1n}, \ldots, L_{nn}\}$ with respect to $\mathcal{B}$ at Z is the $nk \times n^2$ matrix $\mathrm{d}L_Z$ defined by

$$\mathrm{d}L_Z = \begin{array}{r} k\left\{\vphantom{\boxed{Z^t}}\right. \\ k\left\{\vphantom{\boxed{Z^t}}\right. \\ \vdots \\ k\left\{\vphantom{\boxed{Z^t}}\right. \end{array} \left[\begin{array}{cccc} \overbrace{\boxed{Z^t}}^{n} & \overbrace{}^{n} & \cdots & \overbrace{}^{n} \\ & \boxed{Z^t} & & \\ & & \ddots & \\ & & & \boxed{Z^t} \end{array}\right] \tag{2.2}$$

where the $k \times n$ block-matrices on the main diagonal are all equal to Z^t and the other off-diagonal $k \times n$ block-matrices are all equal to 0. We prove the theorem by considering the two cases where either $n \geqslant k$ or $n < k$.

(i) *Case $n \geqslant k$*. Write Z as

$$\begin{array}{r} k\left\{\vphantom{\begin{array}{c}a\\b\end{array}}\right. \\ n-k\left\{\vphantom{\begin{array}{c}a\\b\end{array}}\right. \end{array} \left[\begin{array}{c} Z_k \\ \\ Z_{n-k} \end{array}\right]$$

and let M_0 denote the subset of all $Z \in M$ such that $\det(Z_k) \neq 0$. Then it can be easily shown that M_0 is an open and dense subset of M. Thus if $Z \in M_0$ then rank $(Z) = k$ and the first k rows of Z are linearly independent and the last $n - k$ rows of Z can be expressed as linear combinations of the first k rows. From the matrix (2.2) it follows that the system $\mathcal{A} = \{L_{11}|_Z, \ldots, L_{k1}|_Z, L_{12}|_Z, \ldots, L_{k2}|_Z, \ldots, L_{1n}|_Z, \ldots, L_{kn}|_Z\}$ constitutes a basis for the tangent space to M at Z. In fact, the matrix of this ordered basis relative to the ordered basis $\mathcal{B}$ at Z is

$$\begin{array}{c} k\left\{ \vphantom{Z_k^t} \right. \\ k\left\{ \vphantom{Z_k^t} \right. \\ \vdots \\ k\left\{ \vphantom{Z_k^t} \right. \end{array} \begin{bmatrix} \overbrace{\boxed{Z_k^t}}^{k} & \overbrace{}^{k} & \cdots & \overbrace{}^{k} \\ & \boxed{Z_k^t} & & \\ & & \ddots & \\ & & & \boxed{Z_k^t} \end{bmatrix}. \tag{2.3}$$

Let $g \in G$ denote the inverse of the matrix Z_k then we have

$$\partial_{\alpha i}\Big|_Z = \sum_{j=1,\dots,k} g_{ji} L_{j\alpha}\Big|_Z \qquad 1 \leqslant \alpha \leqslant n \qquad 1 \leqslant i \leqslant k. \tag{2.4}$$

Thus the matrix of the ordered basis $\mathcal{B}$ relative to the ordered basis $\mathcal{A}$ is

$$\begin{bmatrix} \boxed{g} & & & \\ & \boxed{g} & & \\ & & \ddots & \\ & & & \boxed{g} \end{bmatrix}. \tag{2.5}$$

If $\mathcal{D}^\omega(M)$ denotes the algebra of all C^ω differential operators on M then every element $D \in \mathcal{D}^\omega(M)$ can be uniquely written as

$$D = \sum_{|(r)| \leqslant s} f_{(r)} \partial^{(r)} \tag{2.6}$$

where $(r) = (r_{11}, \dots, r_{nk})$ is a multi-index of integers $r_{ij} \geqslant 0$, $|(r)| = r_{11} + \cdots + r_{nk}$, $\partial^{(r)} = \partial_{11}^{r_{11}} \dots \partial_{nk}^{r_{nk}}$, and $f \in C^\omega(M)$ (see, e.g., [Va], p 6). Define a filtration of $\mathcal{D}^\omega(M)$ by setting

$$\mathcal{D}_s^\omega(M) = \left\{ \sum_{|(r)| \leqslant s} f_{(r)} \partial^{(r)} \right\} \qquad s \geqslant 0.$$

The smallest integer $s \geqslant 0$ such that $D \in \mathcal{D}_s^\omega(M)$ is called the degree of D. An element D is called *homogeneous of degree* s if D is of degree s and $D \notin \mathcal{D}_{s-1}^\omega(M)$. By induction on degrees, using the fact that

$$g_{ji} L_{j\alpha}(g_{uv} L_{u\beta}) = g_{ji} g_{uv} L_{j\alpha} L_{u\beta} + g_{ji}(L_{j\alpha}(g_{uv})) L_{u\beta} \tag{2.7}$$

and equation (2.4), it follows that a monomial $\partial_{\alpha 1}^{r_{\alpha 1}}\partial_{\alpha 2}^{r_{\alpha 2}}\ldots\partial_{\alpha k}^{r_{\alpha k}}$, $1 \leqslant \alpha \leqslant n$, can be expressed as a sum of terms of the form

$$\prod_{i=1,\ldots,k}\binom{r_{\alpha i}}{(r_{\alpha i})_1\ldots(r_{\alpha i})_k} g_{1i}^{(r_{\alpha i})_1}\ldots g_{ki}^{(r_{\alpha i})_k} L_{1\alpha}^{S_{\alpha 1}}\ldots L_{k\alpha}^{S_{\alpha k}} + \text{ terms of lower degrees in } L_{\mu\nu} \quad (2.8)$$

where

$$\binom{r_{\alpha i}}{(r_{\alpha i})_1\ldots(r_{\alpha i})_k}$$

denotes a multinomial coefficient and $s_{\alpha j} = (r_{\alpha 1})_j + \cdots + (r_{\alpha k})_j$, $1 \leqslant j \leqslant k$, with $s_{\alpha 1} + \cdots + s_{\alpha k} = r_{\alpha 1} + \cdots + r_{\alpha k}$. It follows that for a fixed multi-index (r) with $|(r)| = s$

$$\partial_{11}^{r_{11}}\partial_{12}^{r_{12}}\ldots\partial_{1k}^{r_{1k}}\partial_{21}^{r_{21}}\ldots\partial_{2k}^{r_{2k}}\ldots\partial_{n1}^{r_{n1}}\ldots\partial_{nk}^{r_{nk}} = \sum_{s_{11}+\cdots+s_{nk}=s} a_{(s)}^{(r)} L_{11}^{s_{11}}\ldots L_{k1}^{s_{1k}} L_{12}^{s_{21}}\ldots L_{k2}^{s_{2k}} L_{1n}^{s_{n1}}\ldots L_{kn}^{s_{nk}}$$
$$+ \text{ terms of lower degree in } L_{\mu\nu} \quad (2.9)$$

where $(s) = (s_{11}, \ldots, s_{nk})$ and $a_{(s)}^{(r)}$ are polynomials in g_{ij} and thus are functions of class $C^{\omega}(M)$.

Now observe that by definition of $\mathcal{D}^{\omega}(M)$ terms of degree zero are just scalars, and equation (2.4) expresses homogeneous terms of degree one in $\partial_{\alpha i}$ in terms of degree one in $L_{\mu\nu}$. Using induction on the degree s of D, with the first step of the induction being equation (2.4), we prove the sth step by assuming the $(s-1)$th step and by using equation (2.1)′ to rewrite the terms of degree $\leqslant s-1$ in equation (2.9) in function of the ordered basis in $L_{\mu\nu}$. It follows that

$$D = \sum_{|(r)|\leqslant s} f_{(r)}\partial^{(r)} = \sum_{|(r)|\leqslant s} h_{(r)} L^{(r)} \quad (2.10)$$

where $L^{(r)} = L_{11}^{r_{11}} L_{21}^{r_{12}}\ldots L_{k1}^{r_{1k}} L_{12}^{r_{21}}\ldots L_{k2}^{r_{2k}}\ldots L_{1n}^{r_{n1}}\ldots L_{kn}^{r_{nk}}$ and $h_{(r)}$ are functions of class $C^{\omega}(M_0)$.

To prove the uniqueness of the coefficients $h_{(r)}$ we again proceed by induction of the degree s of D. For $s = 1$ we have

$$D = c\mathbf{1} + \sum_{\alpha,i} f_{\alpha i}\partial_{\alpha i} \qquad 1 \leqslant \alpha \leqslant n \qquad 1 \leqslant i \leqslant k$$

where $\mathbf{1}$ is the evaluation map defined by

$$\mathbf{1}_Z(f) = f(Z) \qquad (\forall Z \in M, \forall f \in C^{\omega}(M))$$

and $c \in \mathbb{C}$. Equation (2.4) implies that

$$\sum_{\alpha,i} f_{\alpha i}\partial_{\alpha i} = \sum_{\alpha,i} f_{\alpha i}\sum_{j=1,\ldots,k} g_{ji} L_{j\alpha}$$
$$= \sum_{\alpha,j}\left(\sum_{i=1,\ldots,k} f_{\alpha i} g_{ji}\right) L_{j\alpha}.$$

Set $h_{\alpha j} = \sum_{i=1,\ldots,k} f_{\alpha i} g_{ji}$ for all $\alpha = 1, \ldots, n$, and $j = 1, \ldots, k$. Then equation (2.1) implies that

$$\sum_{\alpha,i} f_{\alpha i}\partial_{\alpha i} = \sum_{\alpha,j} h_{\alpha j}\sum_{i=1,\ldots,k} Z_{ji}\partial_{\alpha i}$$
$$= \sum_{\alpha,i}\left(\sum_{j=1,\ldots,k} Z_{ji} h_{\alpha j}\right)\partial_{\alpha i}.$$

From the uniqueness of the coefficients $f_{\alpha i}$ it follows that

$$\sum_{j=1,\dots,k} Z_{ji} h_{\alpha j} = f_{\alpha i} \qquad 1 \leqslant \alpha \leqslant n \qquad 1 \leqslant i \leqslant k. \tag{2.11}$$

Equation (2.11) is a system of nk linear equations in nk unknowns $h_{\alpha j}$, $1 \leqslant \alpha \leqslant n$, $1 \leqslant j \leqslant k$. The coefficient matrix of this system is precisely given in equation (2.3). By assumption this matrix is invertible, therefore the coefficients $h_{\alpha j}$ are uniquely determined. Thus step one of the induction holds. For step s we observe that it is sufficient to prove the uniqueness of the coefficients $h_{(r)}$ of the homogeneous elements of the highest degree s. Indeed, suppose that any differential operator of degree $\leqslant s-1$ can be uniquely written in terms of the ordered basis $L^{(r)}$ with coefficients of class $C^{\omega}(M)$, then using equation $(2.1)'$ to rewrite the remainder in equation (2.9) in terms of the ordered basis $L^{(r)}$ of degree $\leqslant s-1$ we obtain the uniqueness of the coefficients $h_{(r)}$ of terms of degree $\leqslant s-1$ of D in equation (2.10). Now let $(s) = (s_{11}, \dots, s_{nk})$ such that $|(s)| = s$ and recall that $L_{\alpha\beta} = \sum_{\ell=1,\dots,k} Z_{\alpha\ell}\partial_{\beta\ell}$. Then a similar calculation as the one used in the derivation of equation (2.9) leads us to the following equation:

$$L_{11}^{s_{11}} \dots L_{k1}^{s_{1k}} L_{12}^{s_{21}} \dots L_{k2}^{s_{2k}} \dots L_{1n}^{s_{n1}} \dots L_{kn}^{s_{nk}} = \sum_{r_{11}+\dots+r_{nk}=s} b_{(r)}^{(s)} \partial_{11}^{r_{11}} \dots \partial_{1k}^{r_{1k}} \partial_{21}^{r_{21}} \dots \partial_{2k}^{r_{2k}} \dots \partial_{n1}^{r_{n1}} \dots \partial_{nk}^{r_{nk}}$$
$$+\text{terms of lower degree in } \partial_{\alpha i} \tag{2.12}$$

where the $b_{(r)}^{(s)}$ are polynomial functions in Z. Now for all multi-indices (r) and (s) such that $|(r)| = |(s)| = s$ let $A = (a_{(s)}^{(r)})$ denote the matrix with row index (s) and column index (r) where $a_{(s)}^{(r)}$ are defined by equation (2.9), and similarly let $B = (b_{(r)}^{(s)})$. Then equations (2.9) and (2.12) imply that

$$\partial^{(r)} = \sum_{|(r')|=s} \left(\sum_{|(s)|=s} b_{(r')}^{(s)} a_{(s)}^{(r)} \right) \partial^{(r')}. \tag{2.13}$$

From the uniqueness of the expression of a differential operator in terms of the ordered basis $\partial^{(r)}$ equation (2.13) implies immediately that $BA = I$, where I is the identity matrix. It follows that A is the inverse of B. In equation (2.10) let D_s denote the homogeneous term of degree s of D then

$$D_s = \sum_{|(r)|=s} f_{(r)} \partial^{(r)} = \sum_{|(s)|=s} h_{(s)} \sum_{|(r)|=s} b_{(r)}^{(s)} \partial^{(r)}$$
$$= \sum_{|(r)|=s} \left(\sum_{|(s)|=s} b_{(r)}^{(s)} h_{(s)} \right) \partial^{(r)}. \tag{2.14}$$

Let $F = (f_{(r)})$ denote the column matrix with row index (r), and similarly set $H = (h_{(s)})$. Equation (2.14) immediately implies $BH = F$, or using the fact that A is the inverse of B we get $H = AF$. The uniqueness of $f_{(r)}$ then implies the uniqueness of $h_{(s)}$ for all multi-indices (s) such that $|(s)| = s$. Thus the induction is complete, and therefore, the proof of the uniqueness of coefficients $h_{(r)}$ in the expression of D in terms of $L^{(r)}$ is also achieved. Since M_0 is open and dense in M we may assume that $h_{(r)} \in C^{\omega}(M)$.

Now if D is a right-invariant differential operator then equation (2.10) implies that

$$\left(\sum_{|(r)|\leqslant s} h_{(r)} L^{(r)} (R(g) f) \right)(Z) = R(g) \left(\sum_{|(r)|\leqslant s} h_{(r)} L^{(r)} f \right)(Z) \tag{2.15}$$

for all $f \in C^{\omega}(M)$, $g \in G$, and $Z \in M$. Since the $L_{\alpha\beta}$ are right-invariant vector fields it follows that the $L^{(r)}$ are right-invariant differential operators. Thus equation (2.15) becomes

$$\sum_{|(r)|\leqslant s} h_{(r)}(Z) (L^{(r)} f)(Zg) = \sum_{|(r)|\leqslant s} h_{(r)}(Zg) (L^{(r)} f)(Zg) \tag{2.16}$$

for all $g \in G$ and all $f \in C^{\omega}(M)$. The uniqueness of the coefficients $h_{(r)}$ implies that

$$h_{(r)}(Z) = h_{(r)}(Zg) \qquad \forall Z \in M \qquad \forall g \in G. \tag{2.17}$$

Since $H = AF$ and $a_{(s)}^{(r)}$ are polynomials in g_{ij} and since locally the last $n-k$ rows of Z can be expressed as linear combinations of the first k rows, it follows that the $f_{(r)}$ can be considered as analytic functions of Z_k alone, and hence each $h_{(r)}$ is an analytic function in $Z_{\alpha i}$, $1 \leqslant \alpha, i \leqslant k$. Condition (2.17) implies that each $h_{(r)}$ is locally constant, and we can patch the neighbourhoods together to show that each $h_{(r)}$ is globally constant since M is connected. This completes the proof of the theorem for this case.

(ii) *Case* $n < k$. To prove the theorem for this case we make use of the special case of the previous one when $n = k$. Then $M = \mathbb{C}^{n\times k}$ is naturally embedded in $\mathbb{C}^{k\times k}$. Write every element of $\mathbb{C}^{k\times k}$ in the form $[\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}]$ with $Z_1 \in M$. Then G acts on $\mathbb{C}^{k\times k}$ via $([\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}], g) \to [\begin{smallmatrix} Z_1 g \\ Z_2 g \end{smallmatrix}]$. From part (i) with $k = n$ we have two bases for $\mathcal{D}(\mathfrak{g})$, and they are related by equations (2.9) and (2.12). For each degree s we choose an ordering of these bases as follows.

The corresponding basis elements $\partial_{11}^{r_n} \ldots \partial_{nk}^{r_{nk}}$ and $L_{11}^{r_{11}} \ldots L_{kn}^{r_{nk}}$ are listed first and the corresponding basis elements $\partial_{11}^{r_{11}} \ldots \partial_{ij}^{r_{ij}} \ldots \partial_{kk}^{r_{kk}}$ and $L_{11}^{r_{11}} \ldots L_{ji}^{r_{ij}} \ldots L_{kk}^{r_{kk}}$ with some exponent $r_{ij} > 0$, $n < i \leqslant k$, are listed last. In both cases the exponents r_{ij} sum up to s. From equations (2.9) and (2.12) we deduce that with this ordering the matrices A and B factored in block matrices as

$$A = \begin{bmatrix} A' & 0 \\ 0 & A'' \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} B' & 0 \\ 0 & B'' \end{bmatrix} \tag{2.18}$$

where A' (resp. B') is the matrix of the first set of basis elements $\partial^{(r)}$ (resp. $L^{(r)}$) in terms of the first set of basis elements $L^{(r)}$ (resp. $\partial^{(r)}$), and similarly for A'' and B''. Thus $A'B' = I$ and $A''B'' = I$.

Now let $D|Z_1 = \sum_{r_{11}+\cdots+r_{nk}\leqslant s} f_{r_{11}\ldots r_{nk}}(Z_1)\partial_{11}^{r_{11}}|Z_1 \ldots \partial_{nk}^{r_{nk}}|Z_1$ be a right-invariant differential operator on $\mathbb{C}^{n\times k}$. Then obviously D is also a right-invariant differential operator on $\mathbb{C}^{k\times k}$; the functions $f_{r_{11}\ldots r_{nk}}$ can be considered as functions on $\mathbb{C}^{k\times k}$ which are independent of the variable Z_2. As in the case (i) we now proceed by induction on the degree s, and we may only consider the case of a homogeneous differential operator D of degree s. Then for $s = 1$ $D = f_{11}\partial_{11} + \cdots + f_{1k}\partial_{1k} + f_{21}\partial_{21} + \cdots + f_{2k}\partial_{2k} + \cdots + f_{n1}\partial_{n1} + \cdots + f_{nk}\partial_{nk}$. By part (i) and our ordering we have

$$\begin{aligned} D = {} & h_{11}L_{11} + h_{12}L_{21} + \cdots + h_{1k}L_{k1} + h_{21}L_{12} + \cdots + h_{2k}L_{k2} + \cdots \\ & + h_{n1}L_{1n} + \ldots + h_{nk}L_{kn} + h_{n+1,1}L_{n+1,1} + \cdots + h_{kk}L_{kk} \end{aligned}$$

where the h_{ij} are scalars. This leads to the following systems of linear equations:

$$[Z^t] \begin{bmatrix} h_{i1} \\ \vdots \\ h_{ik} \end{bmatrix} = \begin{bmatrix} f_{i1} \\ \vdots \\ f_{ik} \end{bmatrix} \qquad 1 \leqslant i \leqslant n, \qquad \text{and} \qquad [Z^t] \begin{bmatrix} h_{j1} \\ \vdots \\ h_{jk} \end{bmatrix} = 0 \qquad n < j \leqslant k. \tag{2.19}$$

Again we may suppose that $Z = [\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}]$ is an invertible $k \times k$ matrix. This forces the $h_{i,n+1}, \ldots, h_{ik}$, $1 \leqslant i \leqslant n$ to be zero because the f_{ik} are independent of Z_2. This also implies that $h_{j1} = h_{j2} = \ldots = h_{jk} = 0$ for $n < j \leqslant k$. Thus D can be written uniquely as $\sum_{i,j=1,\ldots,n} h_{ij}L_{ji}$ where the h_{ij} are constants. Now suppose the theorem holds for differential operators of degree $\leqslant s-1$, and consider $D = \sum_{r_{11}+\cdots+r_{nk}=s} f_{r_{11}\ldots r_{nk}} \partial_{11}^{r_{11}} \ldots \partial_{nk}^{r_{nk}}$, a homogeneous right-invariant differential operator of degree s. With our new ordering equation (2.14) implies that

$$B'H' = F \qquad \text{and} \qquad B''H'' = 0 \tag{2.20}$$

where H' is the column matrix with entries $h_{r_{11}\ldots r_{nk}}$ and H'' is the column matrix with entries equal to the remaining $h_{r_{11}\ldots r_{ij}\ldots r_{kk}}$, $r_{ij} > 0$ and $i > n$. Since B'' is invertible H'' equals 0. Since for $j > n$ $L_{ji} = \sum_{\ell=1}^{k} Z_{j\ell}\partial_{i\ell}$, $1 \leqslant i \leqslant n$, depends on the variable Z_2, and a formula analogous to (2.8) expressing $L_{1i}^{s_{i1}} L_{2i}^{s_{i2}} \ldots L_{ki}^{s_{ik}}$ in terms of $\partial_{i1}^{r_{i1}} \ldots \partial_{ik}^{r_{ik}}$ implies that whenever the exponent s_{ij} in the monomial $L_{1i}^{s_{i1}} \ldots L_{ji}^{s_{ij}} \ldots L_{ki}^{s_{ik}}$ is positive the corresponding coefficient $b_{(r)}^{(s)}$ in equation (2.12) must depend on the variable Z_2. If we order the bases $\{\partial^{(r)}\}$ and $\{L^{(r)}\}$ in such a way that the basis elements involving $\partial_{11}^{r_{11}} \ldots \partial_{ij}^{r_{ij}} \ldots \partial_{nk}^{r_{nk}}$ and $L_{11}^{s_{11}} \ldots L_{ji}^{s_{ij}} \ldots L_{kn}^{s_{nk}}$ with $j > n$ and $s_{ij} > 0$ (the lexicographic ordering for example) then the equation $B'H' = F$ implies that the last columns of the matrix $B'(j > n)$ involve Z_2 and the corresponding last rows of H' must be zero since F does not depend on Z_2. Thus we have shown that

$$D = \sum_{r_{11}+\cdots+r_{nn}=s} h_{r_{11}\ldots r_{nn}} L_{11}^{r_{11}} \ldots L_{ji}^{r_{ij}} \ldots L_{nn}^{r_{nn}}$$

where the coefficients $h_{r_{11}\ldots r_{nn}}$ are scalars. This completes the proof of part (ii) and hence of the theorem as well. □

Remark 2.2.

(i) If we consider the symmetrized form of the ordered monomials $L_{\alpha_1\beta_1} \ldots L_{\alpha_s\beta_s}$, i.e., $\sum_{\sigma\in\sum_s} L_{\alpha_{\sigma(1)}\beta_{\sigma(1)}} \ldots L_{\alpha_{\sigma(s)}\beta_{\sigma(s)}}$ where $\sum_s$ is the symmetric groups of index s, then it can be easily shown that the coefficients $h_{(r)}$ in equation (2.10) are symmetric. Most of the algebraic proofs of PBW follow this direction. Our proof for the case $n = k$ follows the proof of PBW in [Go].

(ii) As we can see from the proof of the theorem the coefficients $b_{(r)}^{(s)}$ of the matrix B are polynomial functions in Z. Taking into account that the coefficients $h_{(r)}$ are scalars it follows from equation $BH = F$ that the coefficients $f_{(r)}$ are polynomials in Z. Thus if $\mathcal{U}(\mathfrak{g})$ (resp. $\mathcal{U}(\mathfrak{g}')$) denotes the universal enveloping algebra of $\mathfrak{g}$ (resp. $\mathfrak{g}'$), whose definition already appeared in [Po], then

$$\mathcal{U}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g}) \qquad \textit{and} \qquad \mathcal{U}(\mathfrak{g}') = \mathcal{D}(\mathfrak{g}').$$

3. An algebraic analogue of Segal's theorem for Bargmann–Segal–Fock spaces

If $Z = (Z_{ij})$ is an element of $\mathbb{C}^{n\times k}$ let $Z = X_{ij} + \sqrt{-1}Y_{ij}$; $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$. If $\mathrm{d}X_{ij}$ (resp. $\mathrm{d}Y_{ij}$) denotes the Lebesgue measure on $\mathbb{R}$, we let $\mathrm{d}Z$ denote the Lebesgue product measure on $\mathbb{C}^{n\times k}$. Define a Gaussian measure μ on $\mathbb{C}^{n\times k}$ by

$$\mathrm{d}\mu\,(Z) = \pi^{-nk} \exp[-\operatorname{Tr}(ZZ^*)]\,\mathrm{d}Z. \tag{3.1}$$

A function $f : \mathbb{C}^{n\times N} \to \mathbb{C}$ is holomorphic square-integrable if it is holomorphic on the entire domain $\mathbb{C}^{n\times k}$ and if $\int_{\mathbb{C}^{n\times k}} |f(Z)|^2\,\mathrm{d}\mu(Z) < \infty$. Let $\mathcal{F}_{n\times k}$ denote the *Bargmann–Segal–Fock* space of all holomorphic square-integrable functions. Endowed with the inner product

$$(f|g) = \int_{\mathbb{C}^{n\times k}} f(Z)\overline{g(Z)}\,\mathrm{d}\mu\,(Z) \qquad f, g \in \mathcal{F}_{n\times k} \tag{3.2}$$

$\mathcal{F}_{n\times k}$ has a Hilbert space structure. It can be easily verified that the inner product defined by equation (3.2) coincides with the following inner product:

$$\langle f, g\rangle = f(D)\overline{g(\bar{Z})}|_{Z=0} \tag{3.3}$$

where $f(D)$ denotes the formal power series obtained by replacing $Z_{\alpha i}$ by the partial derivatives $\partial_{\alpha i}(1 \leqslant \alpha \leqslant n, 1 \leqslant i \leqslant k)$. Note that the subspace $P_{n\times k}$ of all polynomial functions on $\mathbb{C}^{n\times k}$

is dense in $\mathcal{F}_{n\times k}$. Later in this section we will use equation (3.3) to define inner products on generalized Bargmann–Segal–Fock spaces when either n or k, or both, tend to infinity.

Let $G_0 = U(k)$ (resp. $G_0' = U(n)$) then it is easy to verify that the representation R_{G_0} (resp. $L_{G_0'}$) defined on $\mathcal{F}_{n\times k}$ by right (resp. left) translation is unitary. Let $L_{G_0'} \otimes R_{G_0}$ denote the joint action of $G_0' \times G_0$ on $\mathcal{F}_{n\times k}$ by

$$[(L_{G_0'} \otimes R_{G_0})(g', g) f](Z) = f((g')^t Z g) \tag{3.4}$$

for all $(g', g) \in G_0' \times G_0$ and $f \in \mathcal{F}_{n\times k}$. Then the representations $L_{G_0'}$ and R_{G_0} are *dual* (see TT1]) and we have the following decomposition:

$$\mathcal{F}_{n\times k} = \sum_{(\lambda)} \oplus \mathcal{I}^{(\lambda)} \tag{3.5}$$

where in equation (3.5) the label (λ) denotes both the signature of an irreducible representation of G_0' and G_0 of the form $(m_1, m_2, \ldots, m_r)$ with $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r \geqslant 0$ non-negative integers and $r = \min(n, k)$. The subspace $\mathcal{I}^{(\lambda)}$ denotes the (λ)-isotypic component, i.e., the direct sum of all irreducible subrepresentations of R_{G_0} (resp. $L_{G_0'}$) that belong to the equivalence class λ_{G_0} (resp. $\lambda_{G_0'}$). Moreover, the restriction of $L_{G_0'} \otimes R_{G_0}$ to $\mathcal{I}^{(\lambda)}$ is irreducible; i.e., $\mathcal{I}^{(\lambda)} \approx V^{\lambda_{G_0'}} \otimes W^{\lambda_{G_0}}$, where $V^{\lambda_{G_0}}$ (resp. $W^{\lambda_{G_0'}}$) is an irreducible G_0-module of class (λ_{G_0}) (resp. G_0'-module of class $(\lambda_{G_0'})$). Moreover, the vector

$$f_{(\lambda)}(Z) = \Delta_1^{m_1-m_2}(A)\Delta_2^{m_2-m_3}(Z)\ldots\Delta^{m_r}(Z) \qquad Z \in \mathbb{C}^{n\times k} \tag{3.6}$$

with $\Delta_i(Z)$, $1 \leqslant i \leqslant r$, the ith principal minor of Z, is the highest weight vector corresponding to the signature $(\lambda_{G_0'}, \lambda_{G_0})$ (see [Ze]). Since G (resp. G') is the complexification of G_0 (resp. G_0') the 'Weyl's unitarian trick' (cf [Va]) implies that the representation $L \otimes R$ of $G' \times G'$ on $\mathcal{I}^{(\lambda)}$ is also irreducible (but of course not unitary) with the same signature $(\lambda_{G'}, \lambda_G)$ and the same highest weight vector $f_{(\lambda)}$. Thus we have a multiplicity-free decomposition of $\mathcal{F}_{n\times k}$ into irreducible $G' \times G$-submodules.

We now consider the Weyl algebra $W_{n\times k}$ of differential operators $\sum_{(r)(s)} c_{(r)(s)} Z^{(r)} \partial^{(s)}$ with polynomial coefficients; $Z = (Z_{\alpha_i})$, $\partial = (\partial_{\alpha i})$, $Z^{(r)} = Z_{11}^{r_{11}} \ldots Z_{n_k}^{r_{nk}}$, $\partial^{(s)} = \partial_{11}^{s_{11}} \ldots \partial^{s_{nk}}$, $c_{(r)(s)} \in \mathbb{C}$, and $c_{(r)(s)} \neq 0$ for at most finitely many multi-indices (r), (s). Then as an algebra $W_{n\times k}$ is generated by the $2nk$ generators $Z_{\alpha i}$, $\partial_{\alpha i}$ subject to the relation

$$[\partial_{\alpha i}, Z_{\beta j}] = \delta_{ij}^{\alpha\beta}, [Z_{\alpha i}, Z_{\beta j}] = [\partial_{\alpha i}, \partial_{\beta j}] = 0$$

where [,] denotes the commutator and $\delta_{ij}^{\alpha\beta}$ is the Kronecker symbol which equals 1 if $\alpha = \beta$ and $i = j$ and 0 otherwise. For more details on Weyl algebras see, e.g., [Di, Eh].

The Weyl algebra $W_{n\times k}$ acts on $P_{n\times k}$, and hence on $\mathcal{F}_{n\times k}$, by formal differentiation. It is easy to verify that the elements $Z^{(r)}\partial^{(s)}$ form a basis for the vector space $W_{n\times k}$. Let $W_{n\times k}^m$ be the subset of all linear combinations of $Z^{(r)}\partial^{(s)}$ such that $|(r)| + |(s)| \leqslant m$. Then clearly $W_{n\times k}^m W_{n\times k}^{m'} \subset W_{n\times k}^{m+m'}$, so $W_{n\times k}$ is a filtered algebra. Set $W_{n\times k}^{-1} = \phi$, for $w \in W_{n\times k}$ the integer $m \geqslant 0$ such that $w \in W_{n\times k}^m$ but $w \notin W_{n\times k}^{m-1}$ is called the *order of* w. then the Weyl algebra $W_{n\times k}$ is the subalgebra of the algebra $\mathcal{D}^w(M)$ of all differential operators defined by equation (2.6) for which $f_{(r)}$ are polynomial functions.

Definition 3.1. *Let $\mathcal{A}$ be a subalgebra of the Weyl algebra $W_{n\times k}$ then the centralizer (or commutant) of $\mathcal{A}$ in $W_{n\times k}$ is defined as the set*

$$\{w \in W_{n\times k} : [w, a] = 0, \qquad \forall a \in \mathcal{A}\}.$$

The centre of $\mathcal{A}$ is defined as the set

$$\mathcal{Z}(\mathcal{A}) = \{w \in \mathcal{A} : [w, a] = 0, \qquad \forall a \in \mathcal{A}\}.$$

Clearly the universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g}')$ (cf remark 2.2 (ii)) are subalgebras of the Weyl algebra $W_{n\times k}$. In this context we have the following corollary 3.2 which is an algebraic analogue of Segal's theorem. Note that corollary 3.2 is essentially theorem 7 (p 553) together with the discussion on Capelli identities (p 564) of [Ho2]. However, our proof is very different from that of [Ho2]. Our proof follows immediately from theorem 2.1, in which $\mathcal{D}(\mathfrak{g}')$ (resp. $\mathcal{D}(\mathfrak{g})$) is by hypothesis the algebra of right (resp. left) invariant differential operators with analytic coefficients (see equation (2.6)) (a local property), but surprisingly theorem 2.1 implies that these coefficients are polynomials (cf remark 2.2 (ii)) (a global property). Whereas [Ho2] already assumes that $\mathcal{D}(\mathfrak{g}')$ and $\mathcal{D}(\mathfrak{g})$ are subalgebras of the Weyl algebra $W_{n\times k}$.

Corollary 3.2. *The universal enveloping algebra $\mathcal{U}(\mathfrak{g}')$ is the centralizer of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ in the Weyl algebra $W_{n\times k}$, and vice-versa. Moreover,*

$$\mathcal{Z}(\mathcal{U}(\mathfrak{g})) = \mathcal{Z}(\mathcal{U}(\mathfrak{g}')) = \mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g}').$$

Proof. From remark 2.2 (ii) we know that $\mathcal{U}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g}') = \mathcal{D}(\mathfrak{g}')$. Now it is easy to show that a differential operator D is right (resp. left) invariant if and only $[R_{ij}, D] = 0$, $\forall i, j = 1, \ldots, k$ (resp. $[L_{\alpha\beta}, D] = 0$, $\forall \alpha, \beta = 1, \ldots n$). Then theorem 2.1 implies that the centralizer of $\mathcal{U}(\mathfrak{g}')$ in $W_{n\times k}$ is $\mathcal{U}(\mathfrak{g})$, and vice versa. By definition the centre of $\mathcal{U}(\mathfrak{g})$ is contained in both $\mathcal{U}(\mathfrak{g})$ and its centralizer in $W_{n\times k}$, so obviously $\mathcal{Z}(\mathcal{U}(\mathfrak{g})) = \mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g}')$. Similarly, $\mathcal{Z}(\mathcal{U}(\mathfrak{g}')) = \mathcal{U}(\mathfrak{g}') \cap \mathcal{U}(\mathfrak{g})$. Thus the proof of the theorem is complete. □

Remark 3.3. *In physics, if G is a symmetry group of some physical system, then the spectra of the G-invariant operators determine the observable quantum numbers of the physical system. Elements of the centre of the universal enveloping algebra of the Lie algebra of G are sometimes called Casimir invariants (cf, e.g., [Ba+Ra] or [Ze]). In [Ba+Ra] Weyl algebras are also called Heisenberg algebras ([Di] gives a slightly different definition of a Heisenberg algebra) and several important theorems relating Heisenberg fields and Lie fields of classical groups are proved.*

Generators of Casimir invariants of classical groups are well known (see, e.g., [Ba+Ra] or [Ze]). However, in the context of corollary 3.2, the minimal number of algebraically independent generators of $\mathcal{Z}(\mathcal{U}(\mathfrak{g})$ (resp. $\mathcal{Z}(\mathcal{U}(\mathfrak{g}'))$ may differ from the one given in [Ba+Ra] or [Ze]. This is not a contradiction but reflects the statement of theorem 2.1 and the fact that we have an explicit decomposition of $\mathcal{F}_{n\times k}$ into isotypic components with double signatures (and hence 'double spectra'). To wit we consider the following example described in the appendix.

From the appendix, as an example, we consider the case $n = 2$ and $k = 3$. Then $\{1, \mathrm{Tr}([L]), \mathrm{Tr}([L]^2)\}$ generates $\mathcal{Z}(\mathcal{U}(\mathfrak{g}'))$. So $\mathrm{Tr}([L]^3)$ can be expressed as a polynomial in 1, $\mathrm{Tr}([L])$, and $\mathrm{Tr}([L]^2)$. From equation (A.3) we see that $\mathrm{Tr}([R]^3)$ can be expressed as a polynomial in 1, $\mathrm{Tr}([R])$ and $\mathrm{Tr}([R]^2)$. Thus, in general, if we fix n, for example, and consider the case k arbitrary with $k \geqslant n$, then to compute the spectra of the representation R on $\mathcal{F}_{n\times k}$ we only need to consider the n Casimir invariants $\mathrm{Tr}([L]^i)$, $0 \leqslant i \leqslant n$. All of these facts are, of course, theoretically evident since corollary 3.2 affirms that $\mathcal{Z}(\mathcal{U}(\mathfrak{g})) = \mathcal{Z}(\mathcal{U}(\mathfrak{g}'))$. Nevertheless they are important in order to understand the generalization of corollary 3.2 to the case of infinite-dimensional unitary and general linear groups. This is what we will turn our attention to.

Representation theory of $U(\infty)$ (and $GL_\infty(\mathbb{C})$) has a long history. Starting with the work of Segal in [Se2], it was thoroughly investigated by Kirillov in [Ki], Stratila and Voiculescu in [St+Vo], Pickrell in [Pi], Ol'shanskii in [Ol1], Gelfand and Graev in [Ge+Gr] and Kac in [Ka], to cite just a few. A more complete list of references can be found in the comprehensive and important work of Ol'shanskii in [Ol2]. For our work we will mostly quote the latter.

Let $G_0(k)$ and $G(k)$ denote the unitary group $U(k)$ and the general linear group $GL(k,\mathbb{C})$, respectively. Then the *inductive limits* $G_0(\infty) = \lim_{\to} G_0(k) = \bigcup_{k=1}^{\infty} G_0(k)$ and $G(\infty) = \lim_{\to} G(k) = \bigcup_{k=1}^{\infty} G(k)$ are defined as follows:

$G(\infty) = \{g = (g_{ij})_{i,j\in\mathbb{N}}:$ all but a finite number of $g_{ij} - \delta_{ij}$ are 0 and g is invertible$\}$

and

$$G_0(\infty) = \{u \in G(\infty) : u^* = u^{-1}\}.$$

Definition 3.4 (G I Ol'shanskii). *A unitary representation of the group $G_0(\infty)$ is called tame if it is continuous in the group topology in which the descending chain of subgroups of the type $\{\begin{pmatrix} 1_k & 0 \\ 0 & * \end{pmatrix}\}$, $k = 1, 2, \ldots$ constitutes a fundamental system of neighbourhoods of the identity 1_∞.*

Consider $G_0(\infty)$ and assume that for each k a unitary representation $(R_k, \mathcal{H}_k)$ of $G_0(k)$ is given and an isometric embedding (of Hilbert spaces) $i_k : \mathcal{H}_k \to \mathcal{H}_{k+1}$ commuting with the action of $G_0(k)$ is given (i.e., $i_k \circ R_k(u) = R_{k+1}(u) \circ i_k\ \forall u \in G_0(k)$). Let $\mathcal{H}_\infty$ denote the Hilbert completion of $\bigcup_{k=1}^{\infty} \mathcal{H}_k$, then there exists uniquely a unitary representation of R_∞ of $G_0(\infty)$ on $\mathcal{H}_\infty$ defined by

$$R_\infty(u)f = R_k(u)f \qquad \text{if} \quad u \in G_0(k) \qquad \text{and} \qquad f \in \mathcal{H}_k.$$

The representation $(R_\infty, \mathcal{H}_\infty)$ is called the *inductive limit* of the sequence $\{(R_k, \mathcal{H}_k)\}$. Then we have the following theorem (see [Ol2] for a proof.).

> If the representations $(R_k, \mathcal{H}_k)$ are all irreducible then the inductive limit $(R_\infty, \mathcal{H}_\infty)$ is also irreducible.

Let $\lambda_{G_0(k)} = (m_1, \ldots, m_k)$, $m_1 \geqslant \ldots \geqslant m_k \geqslant 0$, $m_k \in \mathbb{N}$, be the signature of an irreducible $G_0(k)$-module $\{\rho_\lambda, V^{\lambda_{G_0(k)}}\}$. In [Ol2] it was shown that

> *All unitary irreducible tame representations of $G_0(\infty)$ are inductive limits of sequences of the form $\{\rho_\lambda, V^{\lambda_{G_0(k)}}\}$, where in each $(\lambda) = (m_1, m_2, \ldots, \ldots)$ m_i are equal to 0 if i is sufficiently large.*

It follows from the 'Weyl's unitarian trick' that all irreducible tame representations of $G(\infty)$ are inductive limits of sequences of the form $\{\rho_\lambda, V^{\lambda_{G(k)}}\}$.

Definition 3.5 (G I Ol'shanskii). *A representation of $G_0(\infty)$ (resp. $G(\infty)$) is called holomorphic if it is a direct sum (of any number) of irreducible tame representations.*

Now consider again the dual action of $G_0'(n) \times G_0(k)$ on the Hilbert space $\mathcal{F}_{n\times k}$. Fix n and let $G_0(n,\infty) = \bigcup_{k=1}^{\infty} G_0(k)$. Obviously the natural embedding $i_k : \mathcal{F}_{n\times k} \hookrightarrow \mathcal{F}_{n\times(k+1)}$ satisfies the $i_k \circ (L_{(G_0')_n} \otimes R_{(G_0)_k})(u', u) = (L_{(G_0')_n} \otimes R_{(G_0')_k}(u', u)) \circ i_k$ for all $(u', u) \in G_0'(n) \times G_0(k)$. Let $\mathcal{F}_{n\times\infty}$ denote the Hilbert completion of $\bigcup_{k=1}^{\infty} \mathcal{F}_{n\times k}$, and let $\{L_{(G_0')_n} \otimes R_{(G_0)_\infty}, \mathcal{F}_{n\times\infty}\}$ be the inductive limit of the sequence $\{L_{(G_0')_n} \otimes R_{(G_0)_k}, \mathcal{F}_{n\times k}\}$. Then we have the following theorem.

Theorem 3.6. *The representation $L_{(G_0')_n} \otimes R_{(G_0)_\infty}$ (resp. $L_{G_n'} \otimes R_{G_\infty}$) on $\mathcal{F}_{n\times\infty}$ is holomorphic and the Hilbert space $\mathcal{F}_{n\times\infty}$ is decomposed into an orthogonal direct sum*

$$\mathcal{F}_{n\times\infty} = \sum_{(\lambda)} \oplus \mathcal{I}_{n\times\infty}^{(\lambda)} \tag{3.7}$$

where in equation (3.7) the label (λ) denotes both the signature of an irreducible representation of $G_0'(n)$ and of $G_0(n,\infty)$ of the form $(m_1, m_2, \ldots, m_n)$ and $(m_1, m_2, \ldots, m_n, 0, 0, \ldots)$, respectively. The restriction of $L_{(G_0')_n} \otimes R_{(G_0)_\infty}$ (resp. $L_{G_n'} \otimes R_{G_\infty}$) to the isotypic component $\mathcal{I}_{n\times\infty}^{(\lambda)}$ is irreducible. Moreover the vector $f_{(\lambda)}$ defined by equation (3.6) but with $Z \in \mathbb{C}^{n\times\infty}$ is the highest weight vector corresponding to the double signature $(\lambda_{G_n'}, \lambda_{G_\infty})$.

Proof. First let us observe that the highest weight vector $f_{(\lambda)}$ of the irreducible $G_0(n)\times G_0(k)$-module $\mathcal{I}^{(\lambda)}_{n\times k}$ remains the same for all $k\geqslant n$. Since $f_{(\lambda)}$ is a cyclic vector for each $\mathcal{I}^{(\lambda)}_{n\times k}$, $k\geqslant n$, it follows that we have the embedding

$$i_k^{(\lambda)}:\mathcal{I}^{(\lambda)}_{n\times k}\hookrightarrow\mathcal{I}^{(\lambda)}_{n\times(k+1)}$$

such that

$$i_k^{(\lambda)}\circ(L_{(G_0')_n}\otimes R_{(G_0)k}|_{\mathcal{I}^{(\lambda)}_{n\times k}})(u',u)=(L_{(G_0')_n}\otimes R_{(G_0)k}|_{\mathcal{I}^{(\lambda)}_{n\times k}})(u',u)\circ i_k^{(\lambda)}$$

for all $(u',u)\in G_0'(n)\times G_0(k)$. Therefore, as an inductive limit of irreducible representations $L_{(G_0')_n}\otimes R_{(G_0)_\infty}|_{\mathcal{I}^{(\lambda)}_{n\times\infty}}$ is irreducible. Since at each stage we have the orthogonal direct sum $\mathcal{F}_{n\times k}=\sum_{(\lambda)}\oplus\mathcal{I}^{(\lambda)}_{n\times k}$ at the limit we must have $\mathcal{F}_{n\times\infty}=\sum_{(\lambda)}\oplus\mathcal{I}^{(\lambda)}_{n\times\infty}$, and thus the proof of the theorem is complete. □

Now let $G_0'(\infty)\times G_0(\infty)=\bigcup_{n=1}^{\infty}G_0(n,\infty)$ and let $\mathcal{F}_{\infty\times\infty}$ denote the Hilbert completion of $\bigcup_{n=1}^{\infty}\mathcal{F}_{n\times\infty}$. Then following the same process as in theorem 3.6 we can prove that $\{L_{(G_0')_\infty}\otimes R_{(G_0)_\infty},\mathcal{F}_{\infty\times\infty}\}$ (resp. $\{L_{G'_\infty}\otimes R_{G_\infty},\mathcal{F}_{\infty\times\infty}\}$) is the inductive limit of the sequence $\{L_{(G_0')_n}\otimes R_{(G_0)_\infty},\mathcal{F}_{n\times\infty}\}$ (resp. $\{L_{G'_n}\otimes R_{G_\infty},\mathcal{F}_{n\times\infty}\}$). Similarly if $(\lambda)=(m_1,m_2,\ldots)$ is the signature of either an irreducible tame representation of $G'(\infty)$ or of $G(\infty)$ with m_i equal to 0 for sufficiently large i, then the restriction of $L_{(G_0')_\infty}\otimes R_{(G_0)_\infty}$ to the (λ)-isotypic component $\mathcal{I}^{(\lambda)}_{\infty\times\infty}$ is irreducible as the inductive limit of the sequence

$$\{L(G_0')_n\otimes R_{(G_0)_\infty}|_{\mathcal{I}^{\infty}_{n\times\infty}},\mathcal{I}^{(\lambda)}_{n\times\infty}\}.$$

Corollary 3.7. *The same conclusions as in theorem 3.6 hold for $G_0'(\infty)\times G_0(\infty)$-modules $\mathcal{F}_{\infty\times\infty}$ and $\mathcal{I}^{(\lambda)}_{\infty\times\infty}$.*

Remark 3.8. *A proof of corollary 3.7 can be found in [Ol2] where it is referred to as a Peter–Weyl theorem for the group $G_0(\infty)$.*

Now preserve the notations of theorem 3.6, fix n and consider the chain $\mathcal{F}_{n\times k}\subset\mathcal{F}_{n\times(k+1)}\subset\cdots\subset\mathcal{F}_{n\times\infty}$. For each pair (n,k) let $\mathfrak{g}(n,k)$ and $\mathfrak{g}'(n,k)$ the Lie algebras generated by the infinitesimal operators defined by equation (2.1). Consider the *inverse* (or *projective*) *limit* (see, e.g., [Ro] for the definition of inverse limits) of the inverse system of modules $\{\mathfrak{g}'(n,j),\psi_i^j\}$ where for each pair (i,j), $i\leqslant j$, the connecting morphism $\psi_i^j:\mathfrak{g}'(n,j)\to\mathfrak{g}'(n,i)$ is the truncation map. For example, if

$$L^j_{\alpha\beta}=\sum_{\ell=1,\ldots,j}Z_{\alpha\ell}\partial_{\beta\ell}\qquad\text{and}\qquad L^i_{\alpha\beta}=\sum_{\ell=1,\ldots,i}Z_{\alpha\ell}\partial_{\beta\ell}\qquad 1\leqslant\alpha,\quad\beta\leqslant n$$

then $\psi_i^j(L^j_{\alpha\beta})=L^i_{\alpha\beta}$. Let $\mathfrak{g}'(n,\infty)\equiv\lim_{\leftarrow}\mathfrak{g}(n,k)$ denote the inverse limit of the system of modules $\{\mathfrak{g}'(n,k),\psi_i^k\}$. Then clearly $\mathfrak{g}'(n,\infty)$ is the Lie algebra defined by the generators

$$L^{n\times\infty}_{\alpha\beta}=\sum_{\ell=1}^{\infty}Z_{\alpha\ell}\partial_{\beta\ell}\qquad 1\leqslant\alpha,\beta\leqslant n \tag{3.8}$$

where in equation (3.8) the $L^{n\times\infty}_{\alpha\beta}$ act on the inductive chain $\mathcal{F}_{n\times k}\subset\mathcal{F}_{n\times(k+1)}\subset\ldots\subset\mathcal{F}_{n\times\infty}$ by formal differentiation. Let $\mathcal{U}(\mathfrak{g}'(n,\infty))$ denote the universal enveloping algebra of $\mathfrak{g}'(n,\infty)$ then the action of $\mathfrak{g}'(n,\infty)$ extends obviously to an action of $\mathcal{U}(\mathfrak{g}'(n,\infty))$ on the inductive chain $\mathcal{F}_{n\times k}\subset\mathcal{F}_{n\times(k+1)}\subset\ldots\subset\mathcal{F}_{n\times\infty}$.

The Lie algebra of the infinitesimal action of $G(n,\infty)$ on $\mathcal{F}_{n\times\infty}$ is defined by the generator

$$R^{n\times\infty}_{i,j}=\sum_{\mu=1,\ldots,n}Z_{\mu i}\partial_{\mu j}\qquad\forall i,j\in\mathbb{N}. \tag{3.9}$$

To define the *dual universal enveloping algebra* to $\mathcal{U}(\mathfrak{g}'(n,\infty))$ is a little more delicate. For this we must generalize first the notion of a Weyl algebra to fit our context.

Definition 3.9. *An element of the generalized Weyl algebra $\tilde{\mathcal{W}}_{n\times\infty}$ is a formal series or sum in the column indices of monomials in the indeterminates 1, $Z_{\alpha i}$ and $\partial_{\beta j}$; $1 \leqslant \alpha, \beta \leqslant n, i, j \in \mathbb{N}$.*

It is clear that $\tilde{\mathcal{W}}_{n\times\infty}$ is an algebra in the formal sense. However, to handle the scalars in the operations of this algebra requires caution. For example, in remark 3.3 with $n = 1$ we can define the elements $\mathrm{Tr}([L]^s)$, $\mathrm{Tr}([R^s])$, $s \geqslant 1$, of $\mathcal{W}_{n\times\infty}$ by

$$\begin{aligned}\mathrm{Tr}([L]^s) &= (L_{11}^{n\times\infty})^s = \left(\sum_{i=1}^{\infty} Z_i \partial_i\right)^s \\ \mathrm{Tr}([R]^s) &= \sum_{i_1,i_s=1}^{\infty} R_{i_1 i_2} R_{i_2 i_3} \ldots R_{i_s i_1} \qquad R_{ij} = Z_i \partial_j \qquad i, j \in \mathbb{N}\end{aligned} \tag{3.10}$$

but the relations (A.3) and (A.6) must be handled with care. To wit we compute

$$\begin{aligned}\mathrm{Tr}([R]^2) &= \sum_{i,j=1}^{\infty} R_{ij} R_{ji} = \sum_{i,j=1}^{\infty} (R_{ii} + Z_i Z_j \partial_j \partial_i) \\ &= (\Sigma)\,\mathrm{Tr}([L]) - \mathrm{Tr}([L]) + (\mathrm{Tr}[L])^2\end{aligned}$$

where the symbol Σ denotes the formal series $\sum_{j=1}^{\infty}$ in the indeterminate 1. With this definition when $n = 1$ equation (A.6) generalizes to

$$\mathrm{Tr}([R]^3) = \mathrm{Tr}([L]^3) + 2(\Sigma - 1)(\mathrm{Tr}([L])^2 + (\Sigma - 1)^2\,\mathrm{Tr}([L]).$$

This can be formalized by letting $P(\Sigma)$ denote the commutative polynomial ring over $\mathbb{C}$ in the indeterminate Σ, then $\tilde{\mathcal{W}}_{n\times\infty}$ becomes an algebra over $P(\Sigma)$. Clearly $\tilde{\mathcal{W}}_{n\times\infty}$ acts on $\mathcal{F}_{n\times\infty}$ by formal differentiation and $\bigcup_{k=1}^{\infty} W_{n\times k}$ is contained in $\tilde{\mathcal{W}}_{n\times\infty}$. Moreover the projection map $p_k : \tilde{W}_{n\times\infty} \to W_{n\times k}$ is defined by truncation. Similarly we can define $\tilde{\mathcal{U}}(\mathfrak{g}(n,\infty))$ as the algebra which consists of formal series or sums generated by the R_{ij} and Σ; $i, j \in \mathbb{N}$. An example of an element of $\tilde{\mathcal{U}}(\mathfrak{g}(n,\infty))$ is $\sum_{i,j=1}^{\infty}(\sum_{\alpha=1}^{n} Z_{\alpha i}\partial_{\alpha j})(\sum_{\beta=1}^{n} Z_{\beta j}\partial_{\beta i})$. Clearly $\bigcup_{k=1}^{\infty}\mathcal{U}(\mathfrak{g}(n,k))$ is contained in $\tilde{\mathcal{U}}(\mathfrak{g}(n,\infty))$. Obviously $\mathcal{U}(\mathfrak{g}'(n,\infty))$ and $\tilde{\mathcal{U}}(\mathfrak{g}(n,\infty))$ are subalgebras of $\tilde{\mathcal{W}}_{n\times\infty}$. In this context we have the following generalization of theorem 3.3.

Theorem 3.10. *The universal enveloping algebra $\mathcal{U}(\mathfrak{g}'(n,\infty))$ is the centralizer of the universal enveloping algebra $\tilde{\mathcal{U}}(\mathfrak{g}(n,\infty))$ in the generalized Weyl algebra $\tilde{\mathcal{W}}_{n\times\infty}$, and vice versa. Moreover, if $Z(\mathcal{U}(\mathfrak{g}'(n,\infty)))$ (resp. $\mathcal{Z}(\tilde{\mathcal{U}}(\mathfrak{g}(n,\infty)))$) denotes the centre of $\mathcal{U}(\mathfrak{g}'(n,\infty))$ (resp. $\tilde{\mathcal{U}}(\mathfrak{g}(n,\infty))$) then $\mathcal{Z}(\mathcal{U}'(\mathfrak{g}'(n,\infty))) = \mathcal{Z}(\tilde{\mathcal{U}}(\mathfrak{g},(n,\infty))) = \mathcal{U}(\mathfrak{g}'(n,\infty)) \cap \tilde{\mathcal{U}}(\mathfrak{g}(n,\infty))$.*

Proof. For each k let p'_k be the projection of the inverse limit $\mathcal{U}(\mathfrak{g}'(n,\infty))$ which is defined by truncation then

$$p'_k(\mathcal{U}(\mathfrak{g}'(n,\infty))) = \mathcal{U}(\mathfrak{g}'(n,k)).$$

Since $\mathcal{U}(\mathfrak{g}'(n,k))$ is the centralizer of $\mathcal{U}(\mathfrak{g}(n,k))$ in $\mathcal{W}_{n\times k}$ and we have the chain $\mathcal{U}(\mathfrak{g}(n,k)) \subset \mathcal{U}(\mathfrak{g}(n,k+1)) \subset \cdots \subset \tilde{\mathcal{U}}(\mathfrak{g}(n,\infty))$ it follows that the centralizer of $\tilde{\mathcal{U}}(\mathfrak{g}(n,\infty)$ in $\tilde{\mathcal{W}}_{n\times\infty}$ is $\mathcal{U}(\mathfrak{g}'(n,\infty))$. The converse can be proved in a similar fashion. The proof of the statement regarding the centres is now obvious. □

Finally, let $\mathfrak{g}(\infty,\infty)$ denote the inverse limit of $\mathfrak{g}(n,\infty)$. Thus $\mathfrak{g}(\infty,\infty)$ is generated by the basis elements $R_{ij}^{\infty} = \sum_{\alpha=1}^{\infty} Z_{\alpha i}\partial_{\alpha j}$; $i, j \in \mathbb{N}$. Let $\mathfrak{g}'(\infty,\infty) = \cup_{n=1}^{\infty}\mathfrak{g}'(n,\infty)$, then $\mathfrak{g}'(\infty,\infty)$ is the Lie algebra generated by the basis elements $L_{\alpha\beta} = \sum_{i=1}^{\infty} Z_{\alpha i}\partial_{\beta i}$; $\alpha, \beta \in \mathbb{N}$. Similarly to Definition 3.8 we define the Weyl algebra $\tilde{\mathcal{W}}_{\infty\times\infty}$ which consists of elements which are formal series or sums in both row and column indices of the monomials in the indeterminates 1, $Z_{\alpha i}$, and $\partial_{\beta j}$; $\alpha, \beta, i, j \in \mathbb{N}$. Thus $\tilde{\mathcal{W}}_{\infty\times\infty}$ is an algebra over the commutative ring $P(\sum)$. The universal enveloping algebras $\mathcal{U}(\mathfrak{g}'(\infty,\infty))$ and $\tilde{\mathcal{U}}(\mathfrak{g}(\infty,\infty))$ are defined similarly. Obviously they are subalgebras of $\tilde{\mathcal{W}}_{\infty\times\infty}$. Let $\mathcal{Z}(\tilde{\mathcal{U}}(\mathfrak{g}(\infty,\infty))$ and $\mathcal{Z}(\mathcal{U}(\mathfrak{g}'(\infty,\infty)))$ denote their respective centres. Then, similarly to theorem 3.10, we have theorem 3.11 as follows.

Theorem 3.11. *The universal enveloping algebra* $\mathcal{U}(\mathfrak{g}'(\infty,\infty))$ *and* $\tilde{\mathcal{U}}(\mathfrak{g}(\infty,\infty))$ *are mutual centralizers in the generalized Weyl algebra* $\tilde{\mathcal{W}}_{\infty\times\infty}$. *Moreover, we have*

$$\mathcal{Z}(\mathcal{U}(\mathfrak{g}'(\infty,\infty))) = \mathcal{Z}(\tilde{\mathcal{U}}(\mathfrak{g}(\infty,\infty))) = \mathcal{U}(\mathfrak{g}'(\infty,\infty)) \cap \tilde{\mathcal{U}}(\mathfrak{g}(\infty,\infty)).$$

Remark 3.12.

(i) *Elements of the centres defined in theorems 3.10 and 3.11 are called generalized Casimir invariants. In [Ol3] a notion of generalized Casimir invariants are defined but it is not clear to us if they have any connection with ours. It can be easily shown that the families* $\{\mathrm{Tr}([L^{n\times\infty}]^s)\}_{s\geqslant 0}$, $\{\mathrm{Tr}([R^{n\times\infty}]^s)\}_{s\geqslant 0}$ *form two bases for the common centre of* $\mathcal{U}(\mathfrak{g}'(n,\infty))$ *and* $\tilde{\mathcal{U}}(\mathfrak{g}(n,\infty))$, *where* $\mathrm{Tr}([L^{n\times\infty}]^s)$ *and* $\mathrm{Tr}([R^{n\times\infty}]^s)$ *are defined, respectively, by*

$$\mathrm{Tr}([L^{n\times\infty}]^s) = \sum_{\alpha_1,\dots,\alpha_s=1} L^{n\times\infty}_{\alpha_1\alpha_2}\dots L^{n\times\infty}_{\alpha_s\alpha_1}$$

and

$$\mathrm{Tr}([R^{n\times\infty}]^s) = \sum_{i_{11},\dots,i_s=1}^{\infty} R^{n\times\infty}_{i_1i_2}\dots R^{n\times\infty}_{i_si_1}.$$

Finally, note that in our process we have fixed n *and let* $k\to\infty$, *and then let* $n\to\infty$, *but if we reverse the roles of* n *and* k *the same conclusions still hold.*

(ii) *These generalized Casimir invariants act on the inductive limits* $\mathcal{F}_{n\times\infty}$ *and* $\mathcal{F}_{\infty\times\infty}$ *by formal differentiation. It can be shown that their spectra satisfy a Chevalley–Racah-type theorem and this fact is used to decompose tensor products of irreducible tame representations of* $U(\infty)$ *in [Ho+TT].*

4. Conclusion

We have shown that a PBW theorem can be generalized to the pair $(GL(n,\mathbb{C}), GL(k,\mathbb{C}))$ and a Theorem by Segal can be generalized for the pairs $(GL(n,\mathbb{C}), GL(k,\mathbb{C}))$, $(GL(n,\mathbb{C}), GL_{n\times\infty}(\mathbb{C}))$ and $(GL_{\infty\times\infty}(\mathbb{C}), GL_{\infty,\infty}(\mathbb{C}))$. We also gave a generalization of the notion of Casimir invariants and they seem to have important applications to physics; especially in the problem of explicit decompositions of tensor products of irreducible tame representations of $U(\infty)$. In part II of this paper we will give the same generalizations to other classical dual pairs.

Appendix

Let [L] (resp. [R]) the matrix with entries $L_{\alpha,\beta}$, $1\leqslant\alpha,\beta\leqslant n$ (resp. R_{ij}; $1\leqslant i,j\leqslant k$) then we have two sets of generators of Casimir invariants $\mathrm{Tr}([L]^s)$, $\mathrm{Tr}([R]^s)$, $s\geqslant 1$, corresponding to the representations L and R, respectively.

For $s=1$ we have

$$\mathrm{Tr}([L]) = \sum_\alpha L_{\alpha\alpha} = \sum_\alpha\left(\sum_i Z_{\alpha i}\partial_{\alpha i}\right) = \sum_i\left(\sum_\alpha Z_{\alpha i}\partial_{\alpha i}\right) = \sum_i R_{ii} = \mathrm{Tr}(R).$$

For $s = 2$ we have

$$\begin{aligned}\mathrm{Tr}([L]^2) &= \sum_{\alpha,\beta} L_{\alpha\beta} L_{\beta\alpha} = \sum_{\alpha,\beta} \left(\sum_i Z_{\alpha_i} \partial_{\beta i} \right) \left(\sum_j \beta_j \partial_{\alpha j} \right) \\ &= \sum_{\alpha,\beta} \left(\sum_i Z_{\alpha i} \partial_{\alpha i} + \sum_{i,j} Z_{\alpha i} Z_{\beta j} \partial_{\beta i} \partial_{\alpha j} \right) \\ &= \sum_{\alpha,\beta} \left(L_{\alpha\alpha} + \sum_{i,j} Z_{\alpha i} Z_{\beta j} \partial_{\beta i} \partial_{\alpha j} \right).\end{aligned}$$

So

$$\mathrm{Tr}([L]^2) = n\,\mathrm{Tr}(L) + \sum_{\alpha,\beta} \sum_{i,j} Z_{\alpha i} Z_{\beta j} \partial_{\beta i} \partial_{\alpha j} \tag{A.1}$$

and

$$\begin{aligned}\mathrm{Tr}([R]^2) &= \sum_{i,j} R_{ij} R_{ji} = \sum_{i,j} \left(\sum_\alpha Z_{\alpha i} \partial_{\alpha j} \right) \left(\sum_\beta Z_{\beta j} \partial_{\beta i} \right) \\ &= \sum_{i,j} \left(\sum_\alpha Z_{\alpha i} \partial_{\alpha i} + \sum_{\alpha,\beta} Z_{\alpha i} Z_{\beta j} \partial_{\alpha j} \partial_{\beta i} \right) \\ &= \sum_{i,j} \left(R_{ii} + \sum_{\alpha,\beta} Z_{\alpha i} Z_{\beta j} \partial_{\alpha j} \partial_{\beta i} \right).\end{aligned}$$

So

$$\mathrm{Tr}([R]^2) = k(\mathrm{Tr}[R]) + \sum_{i,j} \sum_{\alpha,\beta} Z_{\alpha i} Z_{\beta j} \partial_{\alpha j} \partial_{\beta i}. \tag{A.2}$$

It follows from equations (A.1) and (A.2) that

$$\begin{aligned}\mathrm{Tr}([R]^2) &= \mathrm{Tr}([L]^2) + (k-n)\,\mathrm{Tr}([L]) \\ \mathrm{Tr}([L]^2) &= \mathrm{Tr}([R]^2) + (n-k)\,\mathrm{Tr}([R]).\end{aligned} \tag{A.3}$$

For $s = 3$ we have, after skipping some tedious computations,

$$\begin{aligned}\mathrm{Tr}([L]^3) = \sum_{\alpha,\beta,\gamma} L_{\alpha\beta} L_{\beta\gamma} L_{\gamma\alpha} &= 2n\,\mathrm{Tr}([L]^2) + (\mathrm{Tr}([L]))^2 - (n^2+1)\,\mathrm{Tr}([L]) \\ &\quad + \sum_{\alpha,\beta,\gamma} \sum_{i,j,\ell} Z_{\alpha i} Z_{\beta j} Z_{\gamma\ell} \partial_{\beta i} \partial_{\gamma j} \partial_{\alpha\ell}\end{aligned} \tag{A.4}$$

and

$$\mathrm{Tr}([R]^3) = 2k\,\mathrm{Tr}([R]^2) + (\mathrm{Tr}([R])^2 - (k^2+1)\,\mathrm{Tr}([R]) + \sum_{i,j,\ell} \sum_{\alpha,\beta,\gamma} Z_{\alpha i} Z_{\beta j} Z_{\gamma\ell} \partial_{\alpha j} \partial_{\beta\ell} \partial_{\gamma i}. \tag{A.5}$$

From equations (A.4) and (A.5) it follows that

$$\begin{aligned}\mathrm{Tr}([R]^3) &= \mathrm{Tr}([L]^3) + 2(k-n)\,\mathrm{Tr}([L]^2) + (k-n)^2\,\mathrm{Tr}([L]) \\ \mathrm{Tr}([L]^3) &= \mathrm{Tr}([R]^3) + 2(n-k)\,\mathrm{Tr}([R]^2) + (n-k)^2\,\mathrm{Tr}([R]).\end{aligned} \tag{A.6}$$

For s large the computations become very complicated, but using induction we can show that $\mathrm{Tr}([R]^s)$ can be expressed as

$$\mathrm{Tr}([R]^s) = \sum_{i=1,\dots,s} c_i\,\mathrm{Tr}([L]^i) \tag{A.7}$$

where the constants c_i are integers depending on n and k, and $c_s = 1$. Thus if we consider the canonical filtration $(\mathcal{U}_s(\mathfrak{g}))_{s\geqslant 0}$ (and similarly $(\mathcal{U}_s(\mathfrak{g}'))_{s\geqslant 0}$) and let $Gr(\mathfrak{g})$ denote the associated graded algebra then the maps

$$\varphi_s : \mathcal{Z}_s(\mathcal{U}(\mathfrak{g}))/\mathcal{Z}_{s-1}(\mathcal{U}(\mathfrak{g})) \longrightarrow \mathcal{Z}_s(\mathcal{U}(\mathfrak{g}'))/\mathcal{Z}_{s-1}(\mathcal{U}(\mathfrak{g}))$$

which send $\mathrm{Tr}([R]^s)$ onto $\mathrm{Tr}([L]^s)$, $s \geqslant 0$, define a vector space isomorphism of the graded algebra associated with $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ onto the graded algebra associated with $\mathcal{Z}(\mathcal{U}(\mathfrak{g}'))$.

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